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APPROXIMATE METHODS OF INVESTIGATING THE FREE VIBRATIONS OF THIN SHELLS

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The linear problem of free vibrations of shells (within the limits of applicability of classical two-dimensional theory) is considered. The possibility of solving this problem by different approximate methods is discussed, where membrane theory and the method of replacing the shell by a plate turn out to be the principle methods. Formal estimates for the asymptotic errors of these methods are deduced.

1. Let us start from the linear equations of a thin elastic shell.

Equilibrium equations

$$\frac{1}{2Eh}(\nabla_\alpha T^{m\alpha} - j_1 b_\alpha{}^m N^\alpha) + j_2 \lambda v^m = 0 \tag{1.1}$$

$$\frac{1}{2Eh}(j_3 b_{\alpha\beta} T^{\alpha\beta} + j_4 \nabla_\alpha N^\alpha) - j_5 \lambda w = 0$$

$$\nabla_\alpha M^{m\alpha} - N^m = 0 \quad \left(\lambda = \frac{\rho\omega^2}{2Eh}\right)$$

Strain displacement formulas

$$j_6 \varepsilon_{mn} = \nabla_m v_n + j_7 b_{mn} w - {}^1/_2 c_{mn} c^{\alpha\beta} \nabla_\alpha v_\beta \tag{1.2}$$

$$\mu_{mn} = \nabla_n \gamma_m - j_8 c_{\alpha m} b_n{}^\alpha \delta, \quad \gamma_m = \nabla_m w - j_8 b_m{}^\alpha v_\alpha$$

$$\delta = -{}^1/_2 c^{\alpha\beta} \nabla_\alpha v_\beta$$

$$(c_{11} = c_{22} = 0, \quad c_{12} = -c_{21} = \sqrt{a_{11}a_{22} - (a_{12})^2})$$

Equations of state

$$T^{mn} = E^{mn\alpha\beta}\varepsilon_{\alpha\beta} + j_1 \frac{h^2}{3} F^{mn\alpha\beta}\mu_{\alpha\beta} \tag{1.3}$$

$$M^{mn} = \frac{h^2}{3}(G^{mn\alpha\beta}\mu_{\alpha\beta} + j_9 H^{mn\alpha\beta}\varepsilon_{\alpha\beta})$$

Here $a_{mn} = \mathbf{r}_m \cdot \mathbf{r}_n$ is the metric tensor of the middle surface, $b_{mn} = \mathbf{n} \cdot \mathbf{r}_{mn}$ is the curvature tensor, $\mathbf{r}$ is the radius-vector of the middle surface, the subscripts on the $\mathbf{r}$ denote differentiation, c_{mn} is the discriminant tensor, ∇_m is the symbol of the covariant derivative (in the metric of the middle surface), λ is a frequency parameter associated with the circular frequency ω, ρ is the shell mass per unit middle surface, h is the shell half-thickness, T^{mn}, N^m, G^{mn} are tensors of the tangential forces, the transverse forces and moments, ε_{mn}, μ_{mn} are tensors of the tangential and bending strains, γ_m, δ are the tensors of the rotation angles relative to the tangential axes and the angle of rotation relative to the normal axis, v_m, w are the tensors of the tangential displacements and the normal displacement, $E^{mn\alpha\beta}$, $F^{mn\alpha\beta}$, $G^{mn\alpha\beta}$, $H^{mn\alpha\beta}$ are physical tensors which can be expressed in a first approximation by the formulas

$$G^{mn\alpha\beta} = E^{mn\alpha\beta} = \frac{2Eh}{1-\nu^2}(a^{\alpha m}a^{\beta n} + \nu c^{\alpha m}c^{\beta n})$$
$$F^{mn\alpha\beta} = H^{mn\alpha\beta} = 0$$

It is considered that the shell performs free harmonic vibrations according to the law $\sin \omega t$ and that the variable t is extracted. The conditional factors j_1, $j_2, \ldots, j_9$ (the subscripts have no tensor value here) are introduced before certain members in (1.1). They will be needed for the subsequent exposition, but must as yet be considered equal to one.

2. Let us intriduce a change in the independent variables by using the equalities (here and below the superscript on η is an exponent)

$$x^m = \eta^p \xi^m, \quad \eta = h/R \tag{2.1}$$

Here x^m are the initial parameters of the selected curvilinear coordinate system, ξ^m are transformed parameters, R is the characteristic radius of curvature of the middle surface, and p is a number dependent on the choice.

The covariant derivative symbol can be represented in the form

$$\nabla_m = \frac{\partial}{\partial x^m} + g_m$$

where g_m is a quantity which does not contain differentiation symbols. Hence, we have by virtue of (2.1)

$$\nabla_m = \eta^{-p}\nabla_m^{\cdot}, \quad \nabla_m^{\cdot} = \frac{\partial}{\partial \xi^m} + \eta^p q_m \tag{2.2}$$

Moreover, let us introduce the following changes of the desired quantities:

$$T^{mn} = \eta^{q+s}\, T_0^{\ mn}, \quad M^{mn} = \eta^{2-2p} M_0^{\ mn}, \quad N^m = \eta^{2-3p} N_0^{\ m} \tag{2.3}$$
$$v_m = \eta^{p+q} v_m^\circ, \quad w = w_0, \quad \gamma_m = \eta^{-p}\gamma_m^\circ, \quad \delta = \eta^q \delta_0$$
$$\varepsilon_{mn} = \eta^{q+s}\varepsilon_{mn}^\circ, \quad \mu_{mn} = \eta^{-2p}\mu_{mn}^\circ, \quad \lambda = \eta^{2r}\lambda_0$$

where q, r, s are as yet arbitrary numbers.

Performing the changes (2.1) - (2.3) and multiplying the transformed equations by powers of η selected in an appropriate manner, we obtain the following system in place of (1.1) - (1.3)

$$\frac{1}{2Eh}\nabla_\alpha^{\cdot} T_0^{m\alpha} - \frac{1}{2Eh}\eta^{2-2p-q-s} b_\alpha^{\ m} N_0^{\ \alpha} + \eta^{2r+2p-s}\lambda_0 v_0^{\ m} = 0 \tag{2.4}$$
$$\frac{1}{2Eh}\eta^k b_{\alpha\beta} T_0^{\alpha\beta} + \frac{1}{2Eh}\eta^{2-4p+k-q-s}\nabla_\alpha^{\cdot} N_0^{\ \alpha} - \eta^{2r+k-q-s}\lambda_0 w_0 = 0$$
$$\nabla_\alpha^{\cdot} M_0^{m\alpha} - N_0^{\ m} = 0$$
$$\eta^s \overset{\circ}{\varepsilon}_{mn} = \nabla_m^{\cdot} v_n^\circ + \eta^{-q} b_{mn} w_0 - {}^1/_2 c_{mn} c^{\alpha\beta}\nabla_\alpha^{\cdot} v_\beta^\circ$$
$$\overset{\circ}{\mu}_{mn} = \nabla_n^{\cdot}\gamma_m^\circ - \eta^{2p+q} c_{\alpha m} b_n^{\ \alpha}\delta, \quad \gamma_m^\circ = \nabla_m^{\cdot} w_0 - \eta^{2p+q} b_m^{\ \alpha} v_\alpha^\circ$$
$$\delta^\circ = -{}^1/_2 c^{\alpha\beta}\nabla_\alpha^{\cdot} v_\beta^\circ$$
$$T_0^{mn} = E^{mn\alpha\beta}\overset{\circ}{\varepsilon}_{\alpha\beta} + \eta^{2-2p-q-s}\frac{R^2}{3}F^{mn\alpha\beta}\overset{\circ}{\mu}_{\alpha\beta}$$
$$M_0^{mn} = \frac{R^2}{3}(G^{mn\alpha\beta}\overset{\circ}{\mu}_{\alpha\beta} + \eta^{2p+q+s}H^{mn\alpha\beta}\overset{\circ}{\varepsilon}_{\alpha\beta})$$

(the factor η^k is introduced in the second of these equalities, where k is still an arbitrary number).

3. Consistent values, i.e., values satisfying all three conditions which are formulated below, are selected for the numbers p, q, r, s, k in (2.4).

CONDITION 1. All the exponents of η should be nonnegative and some of them should equal zero.

If condition 1 is satisfied, then we can put $\eta = 0$ in (2.4) and can pass over to the limit system whose structure depends on the selection of p, q, r, s, k. Taking this into account, we formulate the remaining two conditions thus:

CONDITION 2. The limit system should be formally consistent i.e., the number of equations in any set extracted from them should not exceed the number of unknowns entering therein.

CONDITION 3. At least one of the inertial terms should be contained in the limit system.

Four versions of consistent values of $p, q, r, s, k,$ which satisfy these conditions are given by the following formulas:

VERSION 1
$$q = r = s = k = 0 \tag{3.1}$$

VERSION 2
$$q = -2p, \quad r = -p, \quad s = 0, \quad k = 0 \tag{3.2}$$

VERSION 3
$$q = 0, \quad r = 1 - 2p, \quad s = 2 - 4p, \quad k = 0 \tag{3.3}$$

VERSION 4
$$q = 0, \quad r = 1 - 2p', \quad k = 4p' - 2, \quad s = 0 \tag{3.4}$$

(The number p remains undetermined here, and it is replaced by p' in version 4 for convenience in the subsequent exposition).

It can be verified that the consistency conditions formulated above are satisfied if the values of p and p' are constrained by the following inequalities:

$$0 \leqslant p < {}^1/_2 \qquad \text{for versions 1, 3} \tag{3.5}$$

$$0 \leqslant p < 1 \qquad \text{for version 2}$$

$$^1/_2 \leqslant p' < 1 \qquad \text{for version 4}$$

4. Let us consider the following to be valid.

PROPOSITION. A certain set of solutions of (1.1)-(1.3), satisfying the appropriate limit system to a known degree of approximation, corresponds to each of the consistent values of p, q, r, s, k from the versions constructed in Sect. 3. This proposition will be discussed in greater detail in Sect. 5.

Let us turn to the limit systems of equations corresponding to the four versions of the consistent values of p, q, r, s, k (Sect. 3). The scheme to derive these systems is the following:

Values of q, r, s, k for a given version are given by (3.1)-(3.4). Hence, all the powers of η in the left sides of (2.4) are considered known in each version to the accuracy of the number p, for which the inequalities (3.5) are valid. Moreover, it can be seen that the factor j_n in (1.1)-(1.3) corresponds to each factor of the form

η^m in (2.4). Hence, all the j_n can be separated into three groups: the group j_{n_1} for which the appropriate powers of η are positive for any allowable values of p (or p'); the group j_{n_2} for which the corresponding powers of η are non-negative but take on a zero value for at least one of the allowable values of p (or p'); the group j_{n_3} for which the appropriate powers of η are zero.

The required limit system can evidently be obtained if we set $j_{n_1} = 0$; $j_{n_3} = 1$, while retaining the factors j_{n_2} by considering them zero when the appropriate powers of η are positive, but equal to one when this exponent is zero. By taking such an approach we obtain the following results.

Solutions which we call quasitransverse integrals correspond to version 1. For them we must set $n_1 = 1,4$; $n_2 = 2, 8, 9$; $n_3 = 3, 5, 6, 7$, and the limit system can be represented by changing the order of writing the equations as

$$\frac{1}{2Eh}\nabla_\alpha T^{m\alpha} + j_2\lambda v^m = 0, \quad \frac{1}{2Eh} b_{\alpha\beta} T^{\alpha\beta} - \lambda w = 0 \tag{4.1}$$

$$T^{mn} = E^{mn\alpha\beta}\varepsilon_{\alpha\beta}, \ \varepsilon_{mn} = \nabla_m v_n + b_{mn} w - {}^1/_2 c_{mn} c^{\alpha\beta}\nabla_\alpha v_\beta \ \{j_4 = \eta^{2-4p}\}$$

$$\mu_{mn} = \nabla_n \gamma_m - j_8 c_{\alpha m} b_n{}^\alpha\ \delta, \quad \gamma_m = \nabla_m w - j_8 b_m{}^\alpha v_\alpha \tag{4.2}$$

$$\delta = -{}^1/_2 c^{\alpha\beta}\nabla_\alpha v_\beta$$

$$M^{mn} = \frac{h^2}{3}(G^{mn\alpha\beta}\mu_{\alpha\beta} + j_9 H^{mn\alpha\beta}\varepsilon_{\alpha\beta}), \quad N^m = \nabla_\alpha M^{m\alpha}$$

(here and below, the least power of η which is discarded as compared to one in deriving these limit equations is indicated in the braces):

On the basis of the proposition in Sect. 4, the deduction can hence be made that the quasitransverse integrals are determined in an initial approximation by dynamical membrane equations. These latter are valid (in the construction of the quasitransverse integrals) for $0 \leqslant p < {}^1/_2$ and if $p > 0$ then additional simplifications can be introduced by setting $j_2 = j_8 = j_9 = 0$. The most essential is the simplification $j_2 = 0$. It refers to (4.1), i.e., to the closed subsystem of differential equations determining the tensors T^{mn}, v_m, w, ε_{mn}, and denotes the possibility of neglecting tangential inertial forces in these equations. Less essential are the simplifications $j_8 = j_9 = 0$ which refer to (4.2), i.e., to formulas permitting determination of the direct effects of the remaining unknowns.

Solutions we call quasitangential integrals correspond to version 2. For them $n_1 = 1,4$; $n_2 = 7$; $n_3 = 2, 3, 5, 6, 8, 9$ and the limit system becomes

$$\frac{1}{2Eh}\nabla_\alpha T^{m\alpha} - \lambda v^m = 0, \quad \varepsilon_{mn} = \nabla_m v_n + j_7 b_{mn} w - \frac{1}{2} c_{mn} c^{\alpha\beta}\nabla_\alpha v_\beta \tag{4.3}$$

$$T^{mn} = E^{mn\alpha\beta}\varepsilon_{\alpha\beta} \ \{j_1 = \eta^2\}$$

$$\frac{1}{2Eh} b_{\alpha\beta} T^{\alpha\beta} - \lambda w = 0 \quad \{j_4 = \eta^{2-2p}\} \tag{4.4}$$

$$\mu_{mn} = \nabla_n \gamma_m - c_{\alpha m} b_n{}^\alpha \delta, \quad \gamma_m = \nabla_m w - b_m{}^\alpha v_\alpha \tag{4.5}$$

$$\delta = -{}^1/_2 c^{\alpha\beta}\nabla_\alpha v_\beta$$

$$M^{mn} = \frac{h^2}{3}(G^{mn\alpha\beta}\mu_{\alpha\beta} + H^{mn\alpha\beta}\varepsilon_{\alpha\beta}), \quad \nabla_\alpha M^{m\alpha} - N^m = 0$$

Again dynamical membrane equations are obtained in which additional simplifications can be introduced under definite circumstances. For $p > 0$ the additional simplification $j_7 = 0$ is allowable. It is quite essential since (4.3) are hence transformed into a closed system in T^{mn}, v_m, ε_{mn} which agrees formally in structure with the dynamical equations of the plane problem of elasticity theory. The remaining unknowns w, γ_m, δ, μ_{mn}, M^{mn}, N^m are determined by direct operations from (4.4) and (4.5).

Note. The agreement between (4.3) and the equations of the plane problem (for $j_7 = 0$) is called formal since the metric of the shell middle surface generally differs from the "plane" metric, which can turn out to be important in selecting methods of integrating the equations. In particular, the introduction of Airy functions by using known formulas becomes invalid. However, this is not essential for the questions under discussion here, and for $j_7 = 0$) equations (4.3) will henceforth be provisionally called the equations of the plane problem.

Solutions which we call Rayleigh-type integrals correspond to version 3. For them $n_1 = 6,9$; $n_2 = 1, 2, 8$; $n_3 = 3, 4, 5, 7$, and the limit equations become

$$\nabla_m v_n + b_{mn} w - {}^1/_2 c_{mn} c^{\alpha\beta} \nabla_\alpha v_\beta = 0 \qquad \{j_6 = \eta^{2-4p}\} \tag{4.6}$$

$$\mu_{mn} = \nabla_n \gamma_m - j_8 c_{\alpha m} b_n{}^\alpha \delta, \quad \gamma_m = \nabla_m w - j_8 b_m{}^\alpha v_\alpha \tag{4.7}$$
$$\delta = -{}^1/_2 c^{\alpha\beta} \nabla_\alpha v_\beta$$

$$M^{mn} = \frac{h^2}{3} G^{mn\alpha\beta} \mu_{\alpha\beta}, \quad \nabla_\alpha M^{m\alpha} - N^m = 0 \quad \{j_9 = \eta^{2-2p}\}$$

$$\frac{1}{2Eh} (\nabla_\alpha T^{m\alpha} - j_1 b_\alpha{}^m N^\alpha) + j_2 \lambda v^m = 0 \tag{4.8}$$

$$\frac{1}{2Eh} (b_{\alpha\beta} T^{\alpha\beta} + \nabla_\alpha N^\alpha) - \lambda w = 0$$

These are equations which determine the pure bending state (in the terminology of [1]) or strain without tension (in the terminology of [2]). Equations (4.6) for a closed subsystem in v_m and w; from (4.7) we find γ_m, δ, μ_{mn}, M^{mn}, N^n by direct operations and the second closed subsystem (4.8) is obtained for the component of the tensor T^{mn}. Equations (4.6) - (4.8) are valid for all values of p within the limits $0 \leqslant p < {}^1/_2$ and if $p > 0$, the additional simplifications $j_1 = j_2 = j_8 = 0$ are allowable. They are not essential since they refer to terms considered known.

Solutions we call integrals with high variability correspond to version 4. For them $n_1 = 1, 2, 8, 9$; $n_2 = 3$; $n_3 = 4, 5, 6, 7$ and the limit equations can be represented thus:

$$\frac{1}{2Eh} (j_3 b_{\alpha\beta} T^{\alpha\beta} + \nabla_\alpha N^\alpha) - \lambda w = 0, \quad \nabla_\alpha M^{m\alpha} - N^m = 0 \tag{4.9}$$

$$\mu_{mn} = \nabla_n \gamma_m, \quad \gamma_m = \nabla_m w, \quad M^{mn} = \frac{h^2}{3} G^{mn\alpha\beta} \mu_{\alpha\beta}$$

$$\{j_8 = j_9 = \eta^{2p'}\}$$

$$\nabla_\alpha T^{m\alpha} = 0, \quad T^{mn} = E^{mn\alpha\beta} \varepsilon_{\alpha\beta} \tag{4.10}$$
$$\varepsilon_{mn} = \nabla_m v_n + b_{mn} w - {}^1/_2 c_{mn} c^{\alpha\beta} \nabla_\alpha v_\beta \qquad \{j_1 = j_2 = \eta^{2-2p'}\}$$

For $j_3 = 1$ this system is the dynamical analog of the equations of the approximate theory of the states of stress and strain with high variability (only normal inertial forces are taken into account). The domain of applicability of this system is determined by the inequalities $1/_2 \leqslant p' < 1$, and for $1/_2 < p'$ the additional simplification $j_3 = 0$ becomes allowable. It is essential since (4.9) are consequently separated into an independent closed subsystem in M^{mn}, N^m, μ_{mn}, γ_m, w which agrees formally (to the accuracy of the properties of the metric) with the equations of transverse plate vibrations. Equations (4.10) remain to determine the remaining sought tensors T^{mn}, ε_{mn}, v_m. The w must be considered a known quantity, and therefore, they are inhomogeneous equations of the static plane problem of elasticity theory.

5. Let us strengthen the proposition used in Sect. 4, and let us consider each of the limit systems corresponding to some version of the consistent values of p, q, r, s, k to have solutions in the domain under consideration, in which all the required quantities marked with a zero will have the same asymptotic order in (2.3), but differentiation with respect to ξ^1, ξ^2 cannot result in a substantial growth in these quantities. Then the transformation (2.1) will be the characteristic scale extension for asymptotic approaches but the number p in them agrees in meaning with the index of variability.

The identical factor η^p is taken in the right side of the first equation in (2.1) for both independent variables. Formally this corresponds to the assumption that the variability in both coordinate directions is identical. However, we shall consider below that such directions may exist in which the variability of the functions desired can be substantially smaller, but the corresponding index of variability θ will have a value less than p. Such directions will be called quasistationary, their corresponding index of variability θ is particular, while the index of variability p (in those cases when it is important to distinguish it from θ) is general.

Note. If the quasistationary directions are superposed on the coordinate directions, then it must be considered in solving the limit equations that the derivative with respect to the corresponding variable will vanish. This will be used for further simplifications of certain limit systems.

The order of the quantities marked with zero in (2.3) is identical. This means that the asymptotic of all the desired quantities is established by equations (2.3): if it is considered that the solution is normalized, so that $w = O(\eta^\circ)$, then the orders of the remaining quantities are determined by the powers of η in the right sides of (2.3).

Let P or Q denote the set of tensors governing the shell state of stress and strain, and if it corresponds to any of the integrals introduced into Sect. 4, then we mark P or Q with an additional subscript in parentheses. Then for a broad class of problems the state of stress and strain of a vibrating shell can be represented in the form

$$P = P_{(m)} + \eta^\varkappa Q_{(n)} \tag{5.1}$$

where P is the complete state of stress and strain, $P_{(m)}$, $Q_{(n)}$ are the above-mentioned specific states of stress and strain, the former of which is called principal while the latter is supplementary.

Here $\varkappa$ is understood to be the number which must be selected so that the following iteration process would possibly satisfy the boundary conditions. Two out of the four boundary conditions of shell theory, which can be selected differently for different kinds of vibrations and will be called the principal boundary conditions, must be satisfied in constructing the initial approximation of the principal state of stress and strain $P_{(m)}$. Residuals which must be removed in constructing the supplementary state of stress and strain $Q_{(n)}$ hence appear in the two remaining boundary conditions (supplementary). Consequently, residuals appear in the principal boundary conditions, which must be eliminated in constructing the first correction to $P_{(m)}$, etc.

A homogeneous boundary value problem (principal) is obtained for the initial approximation $P_{(m)}$, during whose solution the eigenvalues $\lambda_{(m)}$ are determined. They are the initial approximation of the required frequency parameter. An inhomogeneous boundary value problem (supplementary) is obtained for the initial approximation of $Q_{(n)}$, in which the frequency parameter must be considered fixed ($\lambda_{(n)} = \lambda_{(m)}$). Henceforth, $P_{(m)}$, $Q_{(n)}$ and λ will always be understood as the initial approximation to these quantities.

Let us assume that a general index of variability, denoted by p' if the state of stress and strain is determined by integrals with high variability, and by p in the remaining cases, can be indicated separately for each $P_{(m)}$ and each $Q_{(n)}$. It is taken into account that $P_{(m)}$ and $Q_{(n)}$ can have quasistationary directions in some part of the domain under consideration. In particular, it is allowed that the shell boundary g for a given kind of vibration should turn out to be a quasistationary line. Hence, we denote the index of variability of the required state of stress and strain along g by θ and we consider that this number is identical for $P_{(m)}$ and $Q_{(n)}$ (it would otherwise be impossible to remove the residuals which $P_{(m)}$ yields in the supplementary boundary conditions by using $Q_{(n)}$).

6. The approach to solving the problems of vibrations theory elucidated in Sect. 5 can be called the method of partitioning the state of stress and strain. It is described in detail in statics in [1]. Let us turn to its applications to dynamics and let us introduce a classification of possible kinds of vibrations, in passing, but without pretending completeness and in agreement with that given in [10]. To be specific, we limit ourselves to the case when the shell edge g is rigidly clamped and we write the appropriate boundary conditions thus

$$v_1 = 0, \quad v_2 = 0; \quad w = 0, \quad \gamma = 0 \quad (\text{on } g) \tag{6.1}$$

(γ is the angle of rotation relative to the tangent to the line g).

QUASITRANSVERSE VIBRATIONS WITH LOW VARIABILITY. Vibrations whose total state of stress is determined by (5.1) for $\varkappa = 0$, $m = 1$, $n = 4$, ($P_{(1)}$ is the principal state of stress and strain defined by the quasitransverse integrals, and $Q_{(4)}$ is the supplementary state of stress defined by integrals with high variability) will be designated thus.

The asymptotics of the quantities in the left sides of the boundary equalities (6.1) is determined by (2.3) and (3.1) for $P_{(1)}$ and by (2.3) and (3.4) for $Q_{(4)}$. Hence (6.1) can be replaced by

$$\overset{\circ}{v}_{1(1)} = -\eta^{p'-p}\overset{\circ}{v}_{1(4)}, \quad \overset{\circ}{v}_{2(1)} = -\eta^{p'-p}\overset{\circ}{v}_{2(4)} \tag{6.2}$$

$$w_{(4)}{}^{\circ} = -w_{(1)}{}^{\circ}, \quad \gamma_{(4)}{}^{\circ} = -\eta^{p'-p}\,\gamma_{(1)}{}^{\circ} \quad (\text{on } g)$$

in which the zeroes have the same value as in (2.3).

By using (2.3), (3.1), (3.4) we obtain $p' = {}^1/_2$ for the general index of variability p' for the supplementary state of stress and strain from the requirement $\lambda_{(n)} = \lambda_{(m)}$. The general index of variability p for $P_{(1)}$ is constrained by the inequalities in the first line of (3.5). It hence follows that all the powers of η written explicitly in (6.2) are positive and these boundary conditions can be expressed approximately by

$$\overset{\circ}{v}_{1(1)} = 0,\quad \overset{\circ}{v}_{2(1)} = 0,\quad w_{(4)}^{\circ} = -w_1^{\circ},\quad \gamma_{(4)}^{\circ} = 0 \qquad (\text{on } g) \tag{6.3}$$

Partition of the complete boundary value problem has been achieved. The principal boundary value problem is to integrate the membrane dynamical equations (4.1) with the homogeneous tangential boundary conditions expressed by the first two equations in (6.3) taken into account. The supplementary boundary value problem is to integrate the dynamical equations of the states of stress with high variability (4.9), and (4.10) with the inhomogeneous nontangential boundary conditions expressed by the last two equalities in (6.3) taken into account. The equality $p' = {}^1/_2$ holds, whereupon $j_3 = 1$ must be set in (4.9). The right side of the third equation in (6.3) should be considered known.

QUASITANGENTIAL VIBRATIONS. Vibrations whose complete state of stress and strain is determined by (5.1) for $\varkappa = 0,\ m = 2,\ n = 5$ ($P_{(2)}$ is the principal state of stress and strain defined by the quasitangential integrals and $Q_{(5)}$ is the supplementary state of stress and strain defined by integrals with high variability) will be designated thus.

We represent (6.1) in the form

$$\overset{\circ}{v}_{1(2)} = -\eta^{p'+p}\ \overset{\circ}{v}_{1(5)},\quad \overset{\circ}{v}_{2(2)} = -\eta^{p'+p}\ v_{(5)},\quad w_{(5)}^{\circ} = -w_{(2)}^{\circ}$$

$$\gamma_{(5)}^{\circ} = -\eta^{p'-p}\ \gamma_1^{\circ}{}_{(2)} \qquad \text{on } (g) \tag{6.4}$$

by using (2.3) and (3.2) for $P_{(2)}$ and (2.3) and (3.4) for $Q_{(5)}$.

By using (2.3), (3.2) and (3.4), we obtain from the condition $\lambda_{(n)} = \lambda_{(m)}$ that p is the general index of variability of $Q_{(5)}$ determined by the formula

$$p' = {}^1/_2\,(1 + p) \tag{6.5}$$

and since p the general index of variability of $P_{(2)}$, is subject to the inequalities in the second line of (3.5), then the exponents of all the explicitly written down powers of η in (6.4) are positive, and these equations can also be replaced approximately by equalities of the form (6.3). This means that a partition of the complete boundary value problem has again been achieved: the principal boundary value problem is to integrate the membrane dynamical equations (4.3), (4.4) with the homogeneous tangential boundary conditions expressed by the first two equalities in (6.3) taken into account, and the supplementary boundary value problem is to integrate the dynamical equations of the states of stress and strain with high variability (4.9) and (4.10) with inhomogeneous, nontangential boundary conditions expressed by the last two equalities in (6.3) taken into account.

For $p > 0$ it is necessary to set $j_7 = 0$ in (4.3) and then, as has already been stated in Sect. 4, to transform these equalities into equations of the plane problem. Moreover, it follows from (6.5) that $p' \geqslant {}^1/_2$. This means that it is generally possible to set $j_3 = 0$ in (4.9) (for $p' > {}^1/_2$), which results in their degeneration into the equations of transverse plate vibrations.

RAYLEIGH - TYPE VIBRATIONS. They are well known and occur in a shell only when the edge supports do not hinder bending of its middle surface. Rigid support of the edge does not leave such freedom and it would be fruitless to study vibrations of Rayleigh-type by starting from conditions (6.1). Without going into this question in detail, let us just note (without proof) that we must set $\varkappa = 1-2p$, $m = 3, n = 4$ in (5.1) for Rayleigh-type vibrations ($P_{(3)}$ and $Q_{(4)}$ are states of stress and strain defined by Rayleigh-type integrals and integrals with high variability, respectively).

QUASITRANSVERSE VIBRATIONS WITH HIGH VARIABILITY. Vibrations for which the principal boundary value problem is to integrate the transverse plate vibration equations (Eqs. (4.9) for $j_3 = 0$) with nontangential boundary conditions taken into account will be understood by this designation. The general index of variability p' for such vibrations should satisfy the inequality $p' > 1/2$ since only then can $j_3 = 0$ be set into (4.9).

Under certain additional conditions, which are expounded below, it can be considered that the complete state of stress is determined only by integrals with high variability. Then, (4.9) and (4.10) can be used in the initial approximation, and since $j_3 = 0$ can be set into (4.9) by assumption, then extraction of the principal problem of integrating the transverse plate vibrations equations with nontangential boundary conditions taken into account, will occur. The supplementary problem will hence be to integrate inhomogeneous static equations of the plane problem (4.10) and to satisfy the tengential boundary conditions.

It follows from the results in [4] (see appendix to [1] also) that (4.10) can have such integrals with high variability for which some line, including the edge g, is quasistationary. Hence, the general index of variability p and the particular index of variability θ in the direction of the edge g should be subjected to the relationship $p = \theta \leqslant p'$ for the supplementary problem. But (4.10) have been derived under the assumption that the general index of variability is identical for all initial quantities. Therefore, the domain of applicability of the approximate method described is constrained by the requirement of validity of the discards resulting in (4.10). From this viewpoint, it is most essential that the term $A = \eta^{2r+2p-s}\lambda_0 v_0^{\ m}$ in the first equality of (2.4) must be discarded. It is here possible to set $s = 0$ ($s \neq 0$ only for Rayleigh-type vibrations). Hence, taking account of the last equality (2.3), we can write $A = \eta^{2p}\lambda v_0^{\ m}$.

Moreover, by assumption λ is determined from the solution of the principal problem. Hence, by virtue of (2.3) and (3.4) it is necessary to consider that $\lambda = O(\eta^{2-4p'})$. In addition, we have $p = \theta$. Therefore, $A \to 0$ as $\eta \to 0$ if and only if the following inequality is satisfied

$$\theta > 2p' - 1 \tag{6.6}$$

by which the domain of applicability of (4.9) and (4.10) is bounded for the approximate investigation of the quasitransverse vibrations with high variability.

If the inequality (6.6) is violated, then it is again necessary to turn to the mode of the solution (5.1) by setting $\varkappa = p + p'$, $m = 6$, $n = 7$ by considering that $P_{(6)}$ is determined by integrals of (4.9) for $j_3 = 0$ and $Q_{(7)}$ is determined by the quasitangential integrals in which $p = 2p' - 1$. The boundary conditions (6.1) reduce by virtue of (2.3), (3.2), (3.4) to the form

$$v_{1(7)}^{\circ} = -v_{1(6)}^{\circ}, \quad v_{2(7)}^{\circ} = -v_{2(6)}^{\circ}, \quad w_{(6)} = -\eta^{p'+p} w_{(7)}^{\circ}, \quad \gamma_{(6)}^{\circ} = -\eta^{2p'} \gamma_{1(7)}^{\circ} \quad (\text{on } g) \qquad (6.7)$$

The exponents are positive in the right sides of the last two equations, and this means that partition is possible even with conditions (6.6) violated. The principal problem remains as before, but the supplementary problem will now be to integrate the equations of the plane dynamical problem with inhomogeneous tangential boundary conditions taken into account (it can be shown that the dynamical equations of the plane problem have integrals for which the edge g is a quasistationary line).

7. Let us discuss the principal boundary value problems clarified in Sect. 6. These boundary value problems can be considered well-studied for quasitangential vibrations and for quasitransverse vibrations with high variability. To the accuracy of the properties of the metric, they agree with boundary value problems of the theory of longitudinal or transverse vibrations of a plate and have nontrivial solutions defining the spectrum of their vibrations. It is certainly necessary to keep only such vibrations in these latter for which the general indices of variability p and p' are bounded by the inequalities in the second and third lines of (3.5).

The principal boundary value problem for quasitransverse vibrations with low variability is to construct nontrivial solutions of the dynamical membrane theory equations which satisfy the tangential boundary conditions, where the tangential inertial forces should be taken into account only for $p = 0$.

The so-called transition lines δ_1, δ_2 on which the principal radii of curvature R_1, R_2 are related to the frequency parameter λ by the equalities $\lambda = 1 / R_1^2$ or $\lambda = 1 / R_2^2$ play an important part for the dynamical membrane equations.

In cases when the shell middle surface contains points of the transition lines for the values of λ under consideration, the solutions of the dynamical membrane equations will undergo infinite discontinuities on δ_1 and δ_2. Hence, solutions of the principal problem for quasitransverse vibrations with low variability cannot exist, i.e., the same situation can hold as in the statical membrane problem for such shells as, say, toroids, when the solutions of the membrane equations must be "revised" by using solutions of the edge-effect-type near the lines of variation in the sign of the middle surface curvature. In statics, however, the transition lines are the exception associated with the geometric specifics of the shell, while in the theory of shell vibrations the appearance of a transition line is the rule and due to the value of the frequency. (In this sense, the exception in the theory of vibrations holds only in the simplest shells, whose principal curvatures are constants).

The procedure for eliminating the discontinuities for shells of revolution is described in [3, 5-7], for instance. General considerations about transition lines can be found in [8].

If the value of λ under consideration is outside the interval

$$\min\left(\frac{1}{R_1^2}, \frac{1}{R_2^2}\right) \leqslant \lambda \leqslant \max\left(\frac{1}{R_1^2}, \frac{1}{R_2^2}\right) \qquad (7.1)$$

i.e., if there are no transition lines on the middle surface, then the principal boundary value problem can have nontrivial solutions. They apparently always exist to the right of the interval (7.1), but cannot be to the left of (7.1) (for shells of revolution the existence conditions for nontrivial solutions of the principal problem have been established in [9] for this case).

8. Turning to a discussion of the supplementary problems, let us start with the quasitransverse vibrations with low variability. In this case, the principal and supplementary states of stress are denoted by $P_{(1)}$ and $Q_{(4)}$, and the supplementary boundary value problem is to integrate the dynamical equations of the states of stress with high variability under inhomogeneous nontangential boundary conditions.

The general index of variability of $Q_{(4)}$ is $1/2$ (Sect. 6). It is hence necessary to start from (4.9) and (4.10) by setting $j_3 = 1$ therein, i.e., by considering that they do not degenerate into the transverse plate vibrations equations. Moreover, the particular index of variability along the edge g is identical for $P_{(1)}$ and $Q_{(4)}$ by assumption, and should be subject to the inequality $0 \leqslant p$, while p, the general index of variability of $P_{(1)}$, should be subject to the inequalities in the first line of (3.5). From this and from the equality $p = 1/2$ we conclude that the edge g is a quasistationary line for $Q_{(4)}$. By using this, local (generally valid only near the edge g) equations of the theory of the states of stress and strain of $Q_{(4)}$ can be constructed. They are obtained from (4.9) and (4.10) as a result of additional simplifications, based on the fact that the directions of a relatively slow change in the desired magnitudes of the state of stress and strain $Q_{(4)}$ are known in advance near the edge g. Without going into the details which can be found in [10], let us present the final result. If the shell middle surface is referred to an arbitrary orthogonal coordinate system (α, β) so that g, would agree with the β- lines, then a system is obtained which is the dynamical analog of the approximate equations of the simple edge effect. Its solution reduces to integrating the equation

$$\frac{h^2}{3(1-\nu^2)} \frac{1}{A^4} \frac{\partial^4 w}{\partial \alpha^4} + \left(\frac{j_3}{R_2'^2} - \lambda\right) w = 0 \tag{8.1}$$

in which A^2 is the coefficient of the first quadratic form and R_2' is the normal radius of curvature of the surface along the β- lines, while j_3 must still be considered equal to one. If the inequality

$$1/R_2'^2 - \lambda > 0 \tag{8.2}$$

is satisfied everywhere on the edge g, then the solution of the supplementary boundary value problem can be constructed exactly as in statics, by introducing only those integrals into the considerations, which damp out with distance from g. Conversely, if the sign of the inequality (8.2) is reversed at all points of g, then (8.1) is known to have a solution which oscillated near g without damping, while the arbitrariness contained in the damped solutions becomes inadquate for compliance with all the conditions of the supplementary boundary value problem. In the terminology of [11], violation of condition (8.2) means irregularity of the degeneration of the complete boundary value problem into the principal boundary value problem (however (8.2) is not the sufficient condition for regularity of the degeneration in the general case).

The states of stress and strain $P_{(2)}$ and $Q_{(5)}$ should be constructed for the quasitangential vibrations, and the supplementary boundary value problem is, as before, to integrate the dynamical equations of the states of stress with high variability under inhomogeneous nontangential conditions. Hence, the relationship (6.5) is valid for p', the general index of variability of $Q_{(5)}$, from which it follows that $p' > 1/2$ generally. Hence, it is necessary to set $j_3 = 0$ in (4.9) and these equalities go over into the equations of transverse plate vibrations, while the supplementary boundary value problem reduces essentially to the computation of the forced steady-state transverse plate vibrations caused by periodically varying edge displacements. This problem has generally a (unique) solution, but the exception is the case when the frequencies of the quasitangential vibrations agree with the eigenvalues of the homogeneous supplementary boundary value problem or, equivalently, with the frequencies of the quasitransverse vibrations with high variability. Such agreement will henceforth be called internal resonance.

The inequality $p' > p$, in whose right side is the general index of variability of $P_{(2)}$, follows from (6.5) for all allowable values of $p\ (p < 1)$. This means that $p' > 0$, i.e., for the supplementary state of stress $Q_{(5)}$, the edge g, as for $Q_{(4)}$, is a quasistationary line. Therefore, the subsystem (4.9) near g can additionally be simplified and a local governing equation can be deduced. It is evidently obtained if we set $j_3 = 0$ in (8.1). Hence, (8.1) goes over into the static equation of the simple edge effect for $\lambda = 0$ and $j_3 = 1$, while for $\lambda \neq 0,\ j_3 = 0$ it is the result of certain simplifications of the transverse plate vibrations equations. Physically it is clear that internal resonance is impossible in the first case, while it is inevitable for certain values of λ in the second. For quasitransverse vibrations with low variability, the members in the parentheses in (8.1) are commensurate and it can be assumed that internal resonance becomes possible if and only if λ exceeds the lower bound of the interval (7.1) on a certain section of the edge.

For quasitransverse vibrations with high variability the supplementary boundary value problem reduces to the solution of the statical plane problem of elasticity theory upon compliance with condition (6.6) and internal resonance is impossible in this case. If condition (6.6) is violated, the supplementary problem reduces to the analysis of the tangential plate vibrations, which will result in internal resonance for definite values of λ.

9. Let us turn to a discussion of the approximate methods of investigating the free shell vibrations by understanding this to be an approximate method to replace the solution of the complete problem by the solution of the principal problem.

The principal boundary value problem for transverse vibrations with low variability and for quasitangential vibrations is to integrate the dynamical membrane equations, and the approximate method to be discussed reduces to the application of membrane theory to investigate the free shell vibrations. Different additional simplifications are allowable in the appropriate equations under known circumstances. Namely, the tangential inertial forces can be discarded for quasitransverse vibrations, and normal displacements can be discarded in the expressions for the tangential strain components for quasitangential vibrations. There hence results that the membrane theory (with compliance with the tangential boundary conditions) can be used formally to construct quasitransverse vibrations with low variability in the frequency parameter lies outside the interval (7.1) and also to construct quasitangential vibrations for any λ and any variability without emerging from the domain of applicability of two-dimensional shell theory.

The second approximate method of investigating free shell vibrations is to construct quasitransverse vibrations with high variability by using the integration of (4.9) with $j_3 = 0$ and satisfying the nontangential boundary conditions. It can provisionally be called the method of replacing the shell by a plate.

In combination, the two approximate methods, membrane theory and analysis of the shell as a plate, enclose a very broad but not exhaustive class of shell vibrations. This class does not contain: quasitransverse vibrations with low variability for which λ is located in the interval (7.1), quasitransverse vibrations with intermediate variability where Eqs. (4.9) and (4.10) must be the basis of the analysis by considering that $j_3 = 1$, and they do not degenerate into the equation of transverse plate vibrations.

The methods proposed are not applicable when internal resonance occurs. This phenomenon is possible for quasitangential vibrations in any range of variation of λ, for quasitransverse vibrations with low variability - for λ located to the right of the upper bound of the interval (7.1), and for quasitransverse vibrations with high variability, in the case when the inequality (6.6) is violated.

Note. Quasitransverse vibrations with intermediate variability are a relatively particular but important form of vibrations. As a rule, they occur with the lowest frequencies. The intermediate index of variability was obtained as $p' = 1/2$ in Sect. 4. However, it must be kept in mind that the investigation proposed does not pretend to be complete. For example, it does not include vibrations for which the quasistationary directions pass along asymptotic lines of the middle surface. There results from [10] that $p' = 1/3$ for $K < 0$ and $p' = 1/4$ for $K = 0$ for such vibrations (K is the curvature of the middle surface).

10. Let us discuss the inaccuracies of the approximate methods under consideration. They are comprised, firstly, of errors due to discarding the supplementary state of stress and strain, and secondly, of the errors admitted in the construction of the principal state of stress and strain. Let us first estimate the errors of the first kind.

By means of (2.3), (3.1), (3.2), (3.4) we obtain estimates for the quantities related to the states of stress and strain $P_{(1)}$, $P_{(2)}$, $P_{(3)}$ introduced in Sect. 5 (it is considered everywhere that w is commensurate with one)

$$v_{\rho(1)} = O(\eta^{p}), \quad \sigma_{(1)}(T) = O(\eta^{-1}), \quad \sigma_{(1)}(M) = O(\eta^{-2p}) \qquad (10.1)$$
$$v_{\rho(2)} = O(\eta^{-p}), \quad \sigma_{(2)}(T) = O(\eta^{-1-2p}), \quad \sigma_{(2)}(M) = O(\eta^{-2p})$$
$$v_{\rho(4)} = O(\eta^{p'}), \quad \sigma_{(4)}(T) = O(\eta^{-1}), \quad \sigma_{(4)}(M) = O(\eta^{-2p'})$$
$$(\rho = 1, 2)$$

Herein the $\sigma(T)$ and $\sigma(M)$ are understood to be stresses due to the tangential forces and the moments, respectively. (The approximate formulas for $Q_{(5)}$ which are needed below, do not differ from the formulas for $Q_{(4)}$ for arbitrary p', but it must be recalled that the meaning of p' for $Q_{(4)}$ and $Q_{(5)}$ is distinct).

The following comparison formulas in the deformability and stress intensity result from (10.1):

$$V_{(4)} = O(V_{(1)}), \quad \Sigma_{(4)} = O(\Sigma_{(1)}) \qquad (10.2)$$
$$V_{(5)} = O(\eta^{p}V_{(2)}), \quad \Sigma_{(5)} = O(\eta^{1+2p-2p'}\Sigma_{(2)}) = O(\eta^{p}\Sigma_{(2)})$$

wherein V and Σ are understood to be the absolute values of the asymptotically principal displacements and stresses, respectively, and the last equality is converted by using (6.5).

Therefore, $P_{(1)}$ and $Q_{(4)}$ are asymptotically equivalent in both the deformability and the stress intensity and, therefore, the membrane analysis of the quasitransverse vibrations with low variability results formally in an error on the order of $O(1)$. The same situation holds for shells with a fixed edge and in statics. The distinction is just that in statics $Q_{(4)}$ always damps out with distance from the edge, and this means that a membrane analysis far from the line of distortion remains true while the damping of $Q_{(4)}$ will be assured for quasitransverse vibrations with low variability, except that the frequency parameter λ is on the left of the interval (7.1).

Note. If $Q_{(4)}$ contains oscillating components, then the membrane analysis retains its value as the first stage of the approximate investigation. In this case it should be supplemented by the construction of the state of stress and strain $Q_{(4)}$ (this is generally a very complex problem, even for solution on a digital computer, but it is solved in elementary fashion by the method of exponential representation for shells of revolution, for instance).

For quasitangential vibrations, $Q_{(5)}$ is secondary in both the deformability and the stress intensity as compared with $P_{(2)}$ for $p > 0$. Therefore, the membrane analysis for quasitangential vibrations with $p > 0$ always yields a vanishingly small error (as $\eta \to 0$), whose asymptotic order is estimated by means of (10.2) and (6.5), i.e., leads to better results in dynamics than in statics.

If the requirement (6.6) is violated, then the term $P_{(6)}$ is discarded in (5.1) in a computation of the quasitransverse vibrations with high variability by the method of replacing the shell by a plate. Taking into account that $P_{(6)}$ is determined by intagrals with high variability and $Q_{(7)}$ by quasitangential integrals, the estimates (10.1) can be used by replacing the subscripts (4) and (2) by (6) and (7) therein, respectively. Hence

$$\eta^{p+p'}V_{(7)} = O(\eta^{p'}V_{(6)}), \quad \eta^{p+p'}\Sigma_{(7)} = O(\eta^{-1+3p'-p}\Sigma_{(6)}) \tag{10.3}$$

The condition of applicability of the method of replacing the shell by a plate is the inequality $p' > 1/2$. Hence, the exponent for η is positive in the right side of the first relation in (10.3). The second relation in (10.3) possesses the same property since $p = 2p' - 1$ (Sect. 6) in the case under consideration, and therefore

$$-1 + 3p' - p = p'$$

Let us turn to errors in determining the principal state of stress and strain associated with the inaccuracies admitted in both the equations and the boundary conditions of the principal boundary value problem.

Estimates of the orders of the components discarded in the equations are written in the braces for the appropriate systems. However, integrals with high variability are to be constructed and the errors therein will be substantially greater than the relative absolute values of the discarded terms. Namely, on the basis of a discussion of this question in [12.] (Ch. 13, Sect. 4), it can be assumed that discarding quantities on the order of $O(\eta^k)$ implies an error on the order of $O(\eta^{k-p})$ in determining integrals with the index of variability p. Hence, by using the inscriptions in the braces in (4.1), (4.3), (4.4) and (4.9), we obtain the following estimates for the error ε':

$$\varepsilon' = O(\eta^{2-5p}), \quad \varepsilon' = O(\eta^{2-3p}), \quad \varepsilon' = O(\eta^{p'}) \qquad (10.4)$$

respectively, for the quasitransverse vibrations with low variability, the quasitangential vibrations, and the quasitransverse vibrations with high variability.

It can be assumed that the error ε'' associated with the inaccuracies in the boundary conditions of the principal boundary value problems have the same orders as the terms discarded in the equalities which express these conditions. Hence, we obtain the following estimates by virtue of (6.2), (6.4), (6.5) and (6.7):

$$\varepsilon'' = O(\eta^{1/2-p}), \quad \varepsilon'' = O(\eta^{1/2+3/2p}), \quad \varepsilon'' = O(\eta^{p'+p} + \eta^{2p'}) \qquad (10.5)$$

(it is taken into account in the first of these that $p' = 1/2$ for quasitransverse vibrations with low variability).

Finally, the error in the principal state of stress and strain is determined as the maximum of two corresponding estimates (10.4), (10.5).

In conclusion, let us note that, without going into details, the order of the corrections $\delta\lambda$ of the frequency parameter can be assumed equal to the order of the greatest of the absolute values of the members discarded in the equations or the boundary conditions of the principal problem. This means that $\delta\lambda = \max(\varepsilon'', \varepsilon''')$ where

$$\varepsilon''' = O(\eta^{2-4p}), \quad \varepsilon''' = O(\eta^{2-2p}), \quad \varepsilon''' = O(\eta^{2-2p'})$$

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